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Intertwining vectors and the connection between critical vertex and SOS models

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Abstract. A new algebraic Bethe ansatz construction for the six-vertex model is formulated using intertwining vectors and local gauge transformations. A new set of eigenvectors and their associated eigenvalue equations are derived. This basis is well suited to connect vertex, SOS and RSOS models. Exact relations on a finite lattice between vertex and SOS models are proven with the help of the intertwining vectors.

1. Introduction

Impressive progress has been realised in recent years on the resolution of integrable models in statistical mechanics (for reviews see [1-4]).

The underlying mathematical structure that allows all these nice constructions is the Yang-Baxter algebra [3, 4]:

$$R(\theta-\theta')[T(\theta)\otimes T(\theta')]=[T(\theta')\otimes T(\theta)]R(\theta-\theta'). \tag{1.1}$$

This algebra goes back to the eight-vertex model solved by Baxter [5, 6]. The main problem is always the construction of transfer matrix eigenvectors and eigenvalues. Very recently a great number of interacting-round-the-face (IRF) models with multistate spins have been exactly solved ([2, 4]). The partition function per site and the order parameters have been explicitly calculated by means of the inversion relation technique and the corner transfer matrix. However, there does not exist for the IRF models a formalism similar to the Bethe ansatz construction. The equations that are solved to find the integrable cases of the IRF model are the so-called star-triangle equations [1]. These equations are a consequence of the Yang-Baxter algebra when it is possible to find a monodromy matrix for the model in question. This is not the case for the IRF model. The algebraic structure (1.1) is not employed and, in consequence, the eigenvalues and eigenvectors of the model are not known, with the exception of the eigenvalue of major modulus in the thermodynamic limit.

As a consequence of this fact, it is not possible to calculate the finite-size corrections for the eigenvalues. These finite-size corrections enable us to evaluate the conformal properties of the model in the critical case.

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However, the IRF models that have been solved recently are 'related' to the symmetric eight-vertex model (Baxter model) and to other multistate exactly soluble vertex models. This connection can be rigorously established between the vertex models and the non-restricted solid-on-solid (SOS) models. But the mathematical equivalence is lacking in the so-called 'restriction procedure' that leads to the more interesting restricted SOS models (RSOS).

If a rigorous mathematical equivalence could be established between vertex and RSOS models, all the known results that have been obtained from the Bethe ansatz construction could be translated to the RSOS case.

It is the aim of this paper to analyse this connection in the more simple situation of the critical case. The study of the critical case is sufficient to find the conformal properties of the model, and the mathematical treatment becomes simpler. In the critical region of the Baxter model, for example, the elliptic parametrisation has a trigonometric limit.

The fundamental objects in the vertex-IRF connection are the intertwining vectors providing the Boltzmann weights of the SOS model from the vertex model R matrix. Intertwining vectors are known for the eight vertex models [4-6] and generalisations [2] but they are singular in the (critical) trigonometric limit. (We recall that the six-vertex model describes the trigonometric (zero-gap) limit of the eight-vertex model.)

In the present paper, we start by deriving a family of (non-singular) intertwining vectors for the six-vertex model. With their help a new set of eigenvectors for the six-vertex model is constructed using an algebraic Bethe ansatz. The corresponding Bethe ansatz equations are found. These results are exposed in § 2 and § 3. Our eigenvectors turn out to be labelled in general by an integer p with $-N/2<p<N/2$, N being the lattice size. When the anisotropy parameter γ in the model fulfils

$$\gamma=\frac{2\pi n}{Q} \qquad n, Q \in \mathbb{Z}$$

then p runs from 0 to $Q-1$. The ground state belongs to the $p=0$ sector.

The systematic use of the intertwining vectors and their properties allows us to derive a precise connection between vertex and SOS partition functions for lattices of arbitrary size $N\times M$ in § 4. Shown there (equations (4.14) and (4.15)) is the precise relation between the partition functions of both models for fixed and periodic boundary conditions (PBC), respectively. The connection is particularly simple (equation (4.26)) when PBC are chosen, except for the spins at the four corners of the lattice. In that case we find that both models have the same central charge since the free energies coincide for large size up to corrections smaller than N^{-2}. We want to stress that the results of § 4 hold for any vertex model possessing orthogonal intertwining vectors and not only for six- or eight-vertex models.

When the intertwining vectors found in § 2 and § 3 are employed to construct the Boltzmann weights of the corresponding SOS model we do not find the critical limit of the eight-vertex SOS model. The obtained statistical weights are in fact the same of the six-vertex model, independent of the integer parameter l that appears in the eight-vertex SOS model [6]. In order to overcome this problem, and taking into account that the critical eight-vertex model has a continuous symmetry U(1) that is not present in the non-critical case, we have introduced a projector operator to obtain new intertwining vectors through a modified definition for them (equation (5.3)). This projector operator selects sectors of different charge in the R matrix of the six-vertex model. In this way, the resulting SOS Boltzmann weights are the same as those of the critical limit of the eight-vertex sos model, with the correct dependence in the integer parameter l. These results are contained in § 5.

2. Intertwining vectors

The intertwining vectors associated with an R-matrix (solution of the Yang–Baxter equations) are defined as solutions of

$$R(u-v)[\phi_1(u)\otimes\phi_2(v)]=\lambda(u-v)[\phi_3(v)\otimes\phi_4(u)] \tag{2.1}$$

where $\lambda[u-v]$ is a c-number function and $\phi_i(u)(1\leqslant i\leqslant q)$ are q-component vectors.

For the six-vertex model ($q=2$) the R matrix reads

$$R(\theta)=\begin{pmatrix}\sin(\theta+\gamma) & 0 & 0 & 0\\ 0 & \sin(\gamma) & \sin(\theta) & 0\\ 0 & \sin(\theta) & \sin(\gamma) & 0\\ 0 & 0 & 0 & \sin(\theta+\gamma)\end{pmatrix} \tag{2.2}$$

where θ is the spectral parameter and γ the anisotropy parameter. Setting

$$\phi_1=\begin{pmatrix}a_1\\a_2\end{pmatrix}\qquad \phi_2=\begin{pmatrix}b_1\\b_2\end{pmatrix}\qquad \phi_3=\begin{pmatrix}c_1\\c_2\end{pmatrix}\qquad \phi_4=\begin{pmatrix}d_1\\d_2\end{pmatrix} \tag{2.3}$$

equation (2.1) yields four linear equations

$$\begin{aligned}&\sin(u-v+\gamma)a_1b_1=\lambda c_1d_1\\&\sin(\gamma)a_1b_2+\sin(u-v)a_2b_1=\lambda c_1d_2\\&\sin(u-v)a_1b_2+\sin(\gamma)a_2b_1=\lambda c_2d_1\\&\sin(u-v+\gamma)a_2b_2=\lambda c_2d_2.\end{aligned} \tag{2.4}$$

Assuming $a_1\neq 0\neq b_1$, $a_2\neq 0\neq b_2$, we find from (2.4) the solvability condition

$$[z\sin(\gamma)+\sin(u-v)][\sin(\gamma)+z\sin(u-v)]=\sin^2(u-v+\gamma)z \tag{2.5}$$

where $z\equiv a_1b_2/(b_1a_2)$. Equation (2.5) gives

$$z=\mathrm{e}^{\pm\mathrm{i}(\theta+\gamma)}. \tag{2.6}$$

Using once more (2.4) and requiring ϕ_1, ϕ_4 to depend only on u and ϕ_2, ϕ_3 only on v yields

$$\begin{aligned}&\phi_1=X^{\pm}(u) &&\phi_2=X^{\pm}(v-\gamma)\\&\phi_3=X^{\pm}(v) &&\phi_4=X^{\pm}(u-\gamma)\end{aligned} \tag{2.7}$$

where

$$X^{\pm}(u)=\begin{pmatrix}a_{\pm}\,\mathrm{e}^{\pm\mathrm{i}u/2}\\ b_{\pm}\,\mathrm{e}^{\mp\mathrm{i}u/2}\end{pmatrix} \tag{2.8}$$

and $a_{\pm}$, $b_{\pm}$ are constant arbitrary parameters. We have then as a general solution of (2.1) for the six-vertex model

$$R(u-v)[X^{\pm}(u)\otimes X^{\pm}(v-\gamma)]=\sin(u-v+\gamma)[X^{\pm}(v)\otimes X^{\pm}(u-\gamma)]. \tag{2.9}$$

There exist two important particular solutions

$$W_1 = \begin{pmatrix} 0 \\ b \end{pmatrix} \qquad \text{and} \qquad W_2 = \begin{pmatrix} a \\ 0 \end{pmatrix}. \tag{2.10}$$

They follow from (2.8) for $a = 0$ and $b = 0$ respectively. The reference state in the usual Bethe ansatz eigenvectors (see [3, 4]) is a tensor product of vectors v (or w). We construct in § 3 eigenvectors of the six-vertex model using the intertwining vectors (2.8).

The tensor products $X^+(u) \otimes X^-(v+\alpha)$ and $X^-(u) \otimes X^+(v+\alpha)$ fulfil for arbitrary α a set of remarkable equations:

$$\begin{aligned} R(u-v)[X^+(u) \otimes X^-(v+\alpha)] \\ &= \sin(\gamma)[X^+(v) \otimes X^-(u+\alpha)] \\ &\quad + \sin(u-v)[X^-(v+\alpha-\gamma) \otimes X^+(u+\gamma)] \end{aligned} \tag{2.11}$$

$$\begin{aligned} R(u-v)[X^-(u) \otimes X^+(v+\alpha)] \\ &= \sin(\gamma)[X^-(v) \otimes X^+(u+\alpha)] \\ &\quad + \sin(u-v)[X^+(v+\alpha-\gamma) \otimes X^-(u+\gamma)]. \end{aligned} \tag{2.12}$$

Since the arguments of the intertwining vectors become shifted by 0 or $\pm\gamma$ through the action of $R(u-v)$, it is natural to define [6, 3]

$$X_l(u) = X^-(u - l\gamma - t) \qquad Y_l(u) = X^+(u + l\gamma + s) \tag{2.13}$$

where s and t are arbitrary parameters. Now (2.9), (2.11) and (2.12) yield

$$\begin{aligned} R(u-v)[X_l(u) \otimes X_{l+1}(v)] &= \sin(u-v+\gamma)[X_l(v) \otimes X_{l+1}(u)] \\ R(u-v)[Y_l(u) \otimes Y_{l-1}(v)] &= \sin(u-v+\gamma)[Y_l(v) \otimes Y_{l-1}(u)] \end{aligned} \tag{2.14}$$

$$R(u-v)[X_l(u) \otimes Y_k(v)] = \sin(\gamma)[X_l(v) \otimes Y_k(u)] + \sin(u-v)[Y_{k-1}(v) \otimes X_{l-1}(u)] \tag{2.15}$$

$$R(u-v)[Y_l(u) \otimes X_k(v)] = \sin(\gamma)[Y_l(v) \otimes X_k(u)] + \sin(u-v)[X_{k+1}(v) \otimes Y_{l+1}(u)]. \tag{2.16}$$

We want to make some remarks before constructing eigenvectors of the six-vertex model. First, the intertwining vectors introduced by Baxter [6] for the eight-vertex model have a singular limit when the elliptic modulus vanishes. This is why we construct our vectors directly from the six-vertex R matrix.

Moreover, (2.9) and (2.14) look almost like eigenvalue equations. Considering the S matrix

$$S(\theta) = PR(\theta)$$

where $P_{ab,cd} = \delta_{ad}\delta_{bc}$, as usual, gives

$$S(u-v)[X^\pm(u) \otimes X^\pm(v-\gamma)] = \sin(u-v+\gamma)[X^\pm(u-\gamma) \otimes X^\pm(v)]. \tag{2.17}$$

Hence, the intertwining vectors self-reproduce when S acts on their tensor product (up to a shift $\pm\gamma$ in their argument).

It must be noticed that in the lattice light-cone approach to vertex models [10] one finds equations with a similar structure to (2.9), (2.11) and (2.12). In this context the $X^{\pm}(u)$ are particle wavefunctions bearing the peculiar self-reproduction property (2.17).

The intertwining vectors introduced above have a natural adjoint

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \tilde{X} \equiv \begin{pmatrix} -x_2 \\ +x_1 \end{pmatrix} \qquad \tilde{x}_\alpha = \varepsilon_{\alpha\beta} x_\beta \qquad \varepsilon_{21} = +1$$

and scalar product

$$X \cdot Y = \tilde{x}_\alpha y_\alpha = \varepsilon_{\alpha\beta} x_\beta y_\alpha = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}. \tag{2.18}$$

Notice that $X \cdot Y = -Y \cdot X$ and $X \cdot X = 0$.

The intertwining vectors $X_l(u)$, $Y_l(v)$ permit to associate an IRF or SOS model to $R(u-v)$ through (2.14)-(2.16).

Define

$$X^{l,l+1}(u) = X_l(u) \qquad X^{l,l-1}(v) = Y_l(v). \tag{2.19}$$

Then (2.14)-(2.16) can be written as

$$R(u-v)[X^{l,m}(u) \otimes X^{m,n}(v)] = \sum_p W(p, n, l, m|u-v)[X^{l,p}(v) \otimes X^{p,n}(u)] \tag{2.20}$$

where $l, m, n, p \in \mathbb{Z}$ but $|l-m| = |m-n| = |l-p| = |p-n| = 1$. The non-trivial coefficients $W(p, n, m, l|\theta)$ read

$$\begin{aligned} &W(l, l\mp 1, l, l\pm 1|\theta) = \sin(\theta+\gamma) \\ &W(l, l\pm 1, l, l\pm 1|\theta) = \sin(\gamma) \\ &W(l\pm 1, l, l\mp 1, l|\theta) = \sin(\theta). \end{aligned} \tag{2.21}$$

As usual we can define as $W(p, n, m, l|\theta)$ the statistical weight of the faces configuration depicted in figure 1. The state of each face being defined by the integers, l, m, n etc. Notice that the weights (2.21) are l independent. Actually, the usual IRF-vertex duality, (see for instance [11]) applied to the six-vertex model gives weights identical to (2.21). This IRF model is not the trigonometric limit of the SOS model obtained from the eight-vertex one through the intertwining vectors of [6]. We shall come back to this issue in § 5.

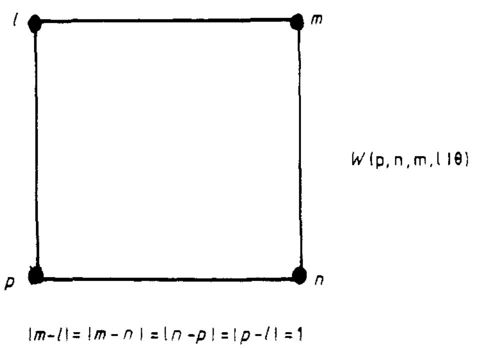

Figure 1. A configuration of site variables round a face of the square lattice and the corresponding Boltzmann weights of the SOS model.

3. Eigenvectors of the six-vertex model from intertwining vectors

Let us now construct eigenvectors of the six-vertex model with the help of the intertwining vectors (2.8). This will be the trigonometric analogy of the construction of [3] for the eight-vertex model. Due to the singular character of the intertwining vectors of [3] in the trigonometric limit we start from our vectors (2.8).

The six-vertex model monodromy matrix reads

$$T_{ab}(\theta)_{\alpha'\gamma} = \sum_{a_1 \ldots a_{N-1}} [t^{(1)}_{a_1 b}(\theta)]_{\alpha_1\gamma_1} \ldots [t^{(N)}_{a a_{N-1}}(\theta)] \tag{3.1}$$

where $[t_{ab}(\theta)]_{\alpha\gamma} = R_{\alpha a}{}^{b\gamma}\,(\theta)$ and $R(\theta)$ was given in (2.2). A family of gauge-transformed monodromy matrices follows by replacing

$$t^{(k)}_{ab}(\theta)_{\alpha\gamma} \to [M^{-1}_{k+l}]_{ac} t^{(k)}_{cd}(\theta)_{\alpha\gamma}[M_{k+l-1}]_{db} \tag{3.2}$$

in (3.1) or, in a more compact notation,

$$t^{(k)}_{\alpha\gamma}(\theta) \to M^{-1}_{k+l} t^{(k)}_{\alpha\gamma}(\theta) M_{k+l-1}. \tag{3.3}$$

Here M is a 2×2 matrix

$$M = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = (X, Y) \tag{3.4}$$

where the vectors X and Y will be determined below. We have

$$M^{-1} = \frac{1}{\Delta}\begin{pmatrix} y_2 & -y_1 \\ -x_2 & x_1 \end{pmatrix} = \frac{1}{X\cdot Y}\begin{pmatrix} -\tilde{Y} \\ \tilde{X} \end{pmatrix} \tag{3.5}$$

where $\Delta = \det M = X\cdot Y$. Therefore, omitting the vertical indices $\alpha\gamma$, the gauge transformation (3.2) yields

$$t \to \hat{t} \equiv M^{-1'} t(\theta) M = \frac{1}{X'\cdot Y'}\begin{pmatrix} -\tilde{Y}'tX, & -\tilde{Y}'tY \\ \tilde{X}'tX, & \tilde{X}'tY \end{pmatrix} \tag{3.6}$$

where $M' \equiv M_{k+l-1}$, $M \equiv M_{k+l}$, and notice that each matrix entry in the RHS of (3.6) is itself a 2×2 matrix acting in the vertical space. For the monodromy matrix associated to a line in the lattice, we find

$$T_{ab}(\theta)_{\alpha|\gamma} \to [M^{-1}_{l+N}]_{ac} T_{cd}(\theta)_{\alpha|\gamma}[M_l]_{db} \equiv T^{(l)}_{ab}(\theta). \tag{3.7}$$

Now, in order to build eigenvectors of the transfer matrix it is very helpful to dispose of a local vacuum. That is, a vector w annihilated by the (2.1) element of t:

$$[\tilde{X}'t(\theta)X]_{\alpha\gamma} w_\gamma = 0 \qquad \text{for} \qquad \alpha = 1, 2. \tag{3.8}$$

This condition can be written as

$$\tilde{x}'_a(R(\theta)[X\otimes w])_{\alpha a} = 0 \qquad \alpha = 1, 2. \tag{3.9}$$

The structure of (3.9) suggests that we choose the vectors x and w as intertwining vectors of $R(\theta)$. Since the eigenvectors of $\tau(\theta)$ can always be made θ independent, it is natural to take

$$\begin{aligned} X &= X^{\pm}(\theta+\alpha) \\ w &= X^{\pm}(\alpha-\gamma) \end{aligned} \tag{3.10}$$

where α is arbitrary. Then (2.9) and (3.9) yield

$$\sin(\theta+\gamma)(X' \cdot X^{\mp}(\theta-\gamma+\alpha))X^{\pm}_a(\alpha)=0 \qquad a=1,2.$$

Hence, a choice for X' such that $X' \cdot X^{\mp}(\theta-\gamma+\alpha)=0$ is

$$X' = X^{\pm}(\theta-\gamma+\alpha). \tag{3.11}$$

Since X and X' correspond to sites k and $k-1$ respectively, we set $\alpha=+(k-l)\gamma-s$ or $+(k+l)\gamma+t$, according to the $\pm$ signs in (3.10) and (3.11):

$$X^l_{k,+}=X_{l+k}(\theta) \qquad X^l_{k,-}=Y_{-l-k}(\theta) \tag{3.12a}$$

or

$$w^l_{k,+}=X_{l+k-1}(0) \qquad w^l_{k,-}=Y_{-l-k+1}(0) \tag{3.12b}$$

where we used (2.13) and l as an arbitrary integer.

Now, the diagonal entries of $t'(\theta)$ acting on the local vacuum w yield

$$[\tilde{Y}'t(\theta)X]_{\alpha\gamma}w_\gamma=\tilde{y}'_\gamma[R(\theta)(X\otimes w)]_{\alpha\gamma}=\sin(\theta+\gamma)(Y'\cdot X')w''_\alpha \tag{3.13}$$

and

$$[\tilde{X}'t(\theta)Y]_{\alpha\gamma}w_\gamma=\tilde{X}'_\gamma[R(\theta)(Y\otimes w)]_{\alpha\gamma} \tag{3.14}$$

where $w''=w_{k+1,\pm}$. In order to recast (3.14) in a form analogous to (2.22), we choose Y to also be an intertwining vector of the type $X+$, so M is non-singular. Looking at (2.14)-(2.16), we can suggest

$$Y=X^+(\theta+[k+l]\gamma+t)=Y_{k+l}(\theta) \tag{3.15a}$$

or

$$Y=X^+(\theta-[k+l]\gamma-s)=X_{-(k+l)}(\theta). \tag{3.15b}$$

We set

$$[\tilde{X}'t(\theta)Y]_{\alpha\gamma}w_\gamma=\sin(\theta)(X'\cdot Y')w'_\alpha \tag{3.16}$$

where w' and Y' correspond to the site $k-1$.

Summarising, (3.7), (3.9), (3.13) and (3.16) give

$$\begin{aligned}\hat{t}^{(k)}_{11}(\theta)w^l_{k,\pm}&=\sin(\theta+\gamma)w^l_{k+1,\pm}\\ \hat{t}^{(k)}_{22}(\theta)w^l_{k,\pm}&=\sin(\theta)w^l_{k-1,\pm}\\ \hat{t}^{(k)}_{21}(\theta)w^l_{k,\pm}&=0\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}&M^+_{k+l}=(X_{l+k}(\theta),\ Y_{l+k}(\theta)) \qquad M^-_{k+l}=(Y_{-(l+k)}(\theta),\ X_{-(k+l)}(\theta))\\ &w^l_{k,+}=X_{l+k+1}(0) \qquad w^l_{k,-}=Y_{-(l+k)+1}(0).\end{aligned} \tag{3.18}$$

Therefore, the families of vectors labelled by $l\in\mathbb{Z}$ and $\pm$

$$\Omega^l_\pm=w^l_{1,\pm}\otimes w^l_{2,\pm}\otimes\ldots\otimes w^l_{N,\pm} \tag{3.19}$$

fulfil the relations

$$\begin{aligned}A^{(l)}(\theta)\Omega^l_\pm&=\sin^N(\theta+\gamma)\Omega^{l-1}_\pm\\ D^{(l)}(\theta)\Omega^l_\pm&=\sin^N(\theta)\Omega^{l+1}_\mp\\ C^{(l)}_{(\theta)}\Omega^l_\pm&=0\end{aligned} \tag{3.20}$$

where, as usual,

$$T^{(l)}_{ab}(\theta)=\begin{pmatrix} A^{(l)}(\theta) & B^{(l)}(\theta) \\ C^{(l)}(\theta) & D^{(l)}(\theta)\end{pmatrix}.$$

In conclusion, we have constructed two infinite families $\Omega_l^{\pm}$ of local vacua with the help of the intertwining vectors of § 2. Both families of reference vectors $\Omega_l^{\pm}$ give equivalent eigenvectors through the algebraic Bethe ansatz construction. In the following we adopt the set Ω_l^{+}.

Let us now show that the intertwining vectors are not only sufficient but also necessary to build the local vacua. Requiring (3.8) for all sites gives

$$\left(\frac{x_{2,k+1}}{x_{1,k+1}}\right)^2+\left(\frac{x_{2,k}}{x_{1,k}}\right)^2-2\cos(\gamma)\,\frac{x_{2,k}}{x_{1,k}}\,\frac{x_{2,k+1}}{x_{1,k}} \tag{3.21}$$

$$\frac{w_k^-}{w_k^+}=\frac{x_{2,k}}{x_{1,k}}\,\mathrm{e}^{\mathrm{i}\gamma/2}. \tag{3.22}$$

Assuming w_k to be θ independent, we find the general solution as

$$\frac{x_{1,k}}{x_{2,k}}=\exp\{\mathrm{i}[\theta-(k+l)\gamma-t]\} \tag{3.23}$$

where t is an arbitrary θ-independent parameter and l an arbitrary integer as before.

In contrast with the usual local vacua of the six-vertex model, w_k is not an eigenvector of $t_{11}^{(k)}$ and $t_{22}^{(k)}$. However, the action of these operators on w_k can be chosen as simple as possible as

$$\hat{t}_{11}(\theta)w_k\sim w_{k+1}\qquad\qquad \hat{t}_{22}(\theta)w_k\sim w_{k-1} \tag{3.24}$$

where the symbol $\sim$ means a proportionality relation. We find, imposing (3.24), the following expressions for the matrix M_n:

$$\begin{aligned} X_k^+&=A\exp\left(\frac{\mathrm{i}}{2}[\theta-(k+l)\gamma-t]\right) \qquad & Y_k^+&=A\exp\left(-\frac{\mathrm{i}}{2}[\theta+(k+l)\gamma+s]\right)\\ X_k^-&=A\exp\left(-\frac{\mathrm{i}}{2}[\theta-(k+l)\gamma-t]\right) \qquad & Y_k^-&=A\exp\left(\frac{\mathrm{i}}{2}[\theta+(k+l)\gamma+s]\right). \end{aligned} \tag{3.25}$$

One recognises here the intertwining vectors (3.12) and (3.15).

Now, in order to build the eigenvectors of the transfer matrix $\tau(\theta)=\tau^{(l)}(\theta)$ it is useful to consider the transformed monodromy matrix [3]

$$T^{(n,l)}(\theta)=M_n^{-1}(\theta)T(\theta)M_l(\theta)=\begin{pmatrix} A_{n,l}(\theta) & B_{n,l}(\theta)\\ C_{n,l}(\theta) & D_{n,l}(\theta)\end{pmatrix}. \tag{3.26}$$

The eigenvectors of $\tau(\theta)$ will be constructed by applying products of $B_{n,s}$ operators on Ω_+^l. Hence, we need their permutation relations with themselves and with the $A_{n,l}$. We find from (3.18) and (3.26)

$$\begin{aligned} A_{n,l}(\theta)&=\Delta^{-1}(\theta)\,\tilde{Y}_n(\theta)\,T(\theta)X_l(\theta)\\ B_{n,l}(\theta)&=\Delta^{-1}(\theta)\,\tilde{Y}_n(\theta)\,T(\theta)\,Y_l(\theta)\\ C_{n,l}(\theta)&=\Delta^{-1}(\theta)\tilde{X}_n(\theta)\,T(\theta)X_l(\theta)\\ D_{n,l}(\theta)&=\Delta^{-1}(\theta)\tilde{X}_nT(\theta)\,Y_l(\theta) \end{aligned} \tag{3.27}$$

where

$$\Delta(\theta) = \det M_l(\theta) = (a_-)(b_+) \exp\left[-\mathrm{i}\left(u + \frac{s-t}{2}\right)\right] - (a_+)(b_-) \exp\left[\mathrm{i}\left(u + \frac{s-t}{2}\right)\right]$$

is l independent. By appropriately projecting the Yang-Baxter algebra (1.1) for $T(\theta)$ on the vectors X_{+l} and Y_{+l} from left and right, we find after some calculations the following permutation relations:

$$B_{k,l+1}(\lambda) B_{k+1,l}(\theta) = B_{k,l+1}(\mu) B_{k+1,l}(\lambda) \tag{3.28}$$

$$A_{k,l}(\lambda) B_{k+1,l-1}(\mu) = \alpha(\lambda-\mu) B_{k,l-2}(\mu) A_{k+1,l-1}(\lambda) - \beta(\lambda-\mu) B_{k,l-2}(\lambda) A_{k+1,l-1}(\mu) \tag{3.29}$$

$$D_{k,l}(\lambda) B_{k+1,l-1}(\mu) = \alpha(\mu-\lambda) B_{k+2,l}(\mu) D_{k+1,l-1}(\lambda) + \beta(\lambda-\mu) B_{k+2,l}(\lambda) D_{k+1,l-1}(\mu) \tag{3.30}$$

where

$$\alpha(\theta) = \frac{\sin(\theta-\gamma)}{\sin(\theta)} \qquad \beta(\theta) = -\frac{\sin(\gamma)}{\sin(\theta)}. \tag{3.31}$$

These permutation relations have the same structure as those arising from [3] in the trigonometric limit except for the coefficient $\beta(\theta)$ that in our case is independent of k and l. This difference is due to the singular character of the expressions of [3] in the trigonometric limit.

Following the generalised Bethe ansatz, we construct the vector

$$\psi_l(\lambda_1, \ldots, \lambda_n) = B_{l+1,l-1}(\lambda_1) \ldots B_{l+n,l-n}(\lambda_n) \Omega_N^{l-n}. \tag{3.32}$$

This is a symmetric function of $\lambda_1, \ldots, \lambda_n$ due to (3.28). Applying to ψ_l the operators $A_{l,l}(\theta)$ and repeatdly using the permutation relations (3.29), we obtain

$$A_{l,l}\psi_l(\lambda_i) = \Lambda_1(\lambda, \lambda_i)\psi_{l-1}(\lambda_i) + \sum_{j=1}^{n} \Lambda_1^j(\lambda, \lambda_i)\psi_{l-1}(\lambda_1, \ldots, \lambda_{j-1}, \lambda, \lambda_{j+1}, \ldots, \lambda_n) \tag{3.33}$$

provided one sets $n = N/2$ as in [3]. After commuting $A_{l,l}$ with all the operators $B_{l+j,l-j}$ in (3.32) we obtain $A_{l+n,l-n}$ which we know how to apply to Ω_N^{l-n} only when $n = N/2$ since, in that case,

$$A_{l+(N/2),l-(N/2)} = A_{(l-(N/2))+N,l-(N/2)} = A^{(l-(N/2))} \tag{3.34}$$

and (3.20) holds. In (3.33)

$$\begin{aligned} \Lambda_1(\lambda, \lambda_i) &= \sin^N(\lambda+\gamma) \prod_{k=1}^{n} \alpha(\lambda-\lambda_k) \\ \Lambda_1^j(\lambda, \lambda_i) &= -\beta(\lambda-\lambda_j) \sin^N(\lambda_j+\gamma) \prod_{\substack{k=1 \\ k\neq j}}^{n} \alpha(\lambda_j-\lambda_k) \end{aligned} \tag{3.35}$$

where $1 \leqslant j \leqslant n$.

In a similar way we have

$$\begin{aligned} D_{l,l}(\lambda)\psi_l(\lambda_i) = {} & \Lambda_2(\lambda, \lambda_i)\psi_{l+1}(\lambda_i) \\ & + \sum_{j=1}^{n} \Lambda_2^j(\lambda, \lambda_1, \ldots, \lambda_n)\psi_{l+1}(\lambda_1, \ldots, \lambda_{j-1}, \lambda, \lambda_{j+1}, \ldots, \lambda_n) \end{aligned} \tag{3.36}$$

where

$$\begin{aligned} &\Lambda_2(\lambda, \lambda_i) = \sin^N(\lambda) \prod_{k=1}^{n} \alpha(\lambda_k - \lambda) \\ &\Lambda_2^j(\lambda, \lambda_i) = \beta(\lambda - \lambda_j) \sin^N(\lambda_j) \prod_{k=1, k\neq j}^{n} \alpha(\lambda_k - \lambda_j) \qquad 1 \leqslant j \leqslant n. \end{aligned} \tag{3.37}$$

ψ_l is not an eigenvector of $\tau(\theta) = A_{l,l} + D_{l,l}$. In order to obtain such eigenvectors we multiply (3.33) and (3.36) by $e^{il\rho}$ where $0 \leqslant \rho < 2\pi$ and sum for all integers. One finds in this way

$$\begin{aligned} \tau(\lambda)\Phi_\rho(\lambda_i) &= [e^{i\rho}\Lambda_1(\lambda, \lambda_i) + e^{-i\rho}\Lambda_2(\lambda, \lambda_i)]\psi_\rho(\lambda_i) \\ &\quad + \sum_{l\in\mathbb{Z}} \sum_{j=1}^{n} e^{il\rho}[e^{i\rho}\Lambda_1^j(\lambda, \lambda_i) + e^{-i\rho}\Lambda_2^j(\lambda, \lambda_i)]\psi_l(\lambda_1, \ldots, \lambda_{j-1}, \lambda, \lambda_{j+1}, \ldots, \lambda_n). \end{aligned} \tag{3.38}$$

Here

$$\Phi_\rho(\lambda_1, \ldots, \lambda_n) \equiv \sum_{l\in\mathbb{Z}} e^{il\rho}\psi_l(\lambda_i). \tag{3.39}$$

As can be seen from (3.38), $\psi_\rho < (\lambda_i)$ is an eigenvector of $\tau(\lambda)$ with eigenvalue

$$\Lambda(\lambda) = e^{i\rho}\Lambda_1(\lambda, \lambda_i) + e^{-i\rho}\Lambda_2(\lambda, \lambda_i)$$

provided the λ_i satisfy the following equations:

$$\left(\frac{\sinh(\mu_j + i\gamma/2)}{\sinh(\mu_j - i\gamma/2)}\right)^N = -e^{-2i\rho} \prod_{k=1}^{n} \frac{\sinh(\mu_j - \mu_k + i\gamma)}{\sinh(\mu_j - \mu_k - i\gamma)} \tag{3.40}$$

where $\mu_j = i\lambda_j + i\gamma/2$. In this way the unwanted terms cancel in (3.37). These Bethe ansatz equations (BAE) coincide with the trigonometric limit of the BAE of [3]. However, the reference states Ω_N^l and gauge-transformed matrices $B_{k,l}(\lambda)$ used to build the eigenvectors in [3] are singular in that limit. It is hard to cancel these singularities in order to find well defined eigenvectors in the six-vertex limit. We preferred in this paper to build the reference vectors and gauge-transformed monodromy matrices free of singularities, working directly in the six-vertex model.

As is already the case in the eight-vertex model [3] the values of ρ in the eigenvectors (3.39) are not arbitrary. They must be determined by requiring that the RHS of (3.38) be non-zero.

Let us now derive the allowed values of ρ in the six-vertex model for arbitrary γ. This is much simpler here, since the eigenvectors contain only exponential functions and not elliptic ones as in the eight-vertex case. In order to obtain the allowed values of ρ it is necessary to analyse the explicit dependence of the right-hand side of (3.38) on the integer parameter l. As can be seen from the expression (3.32), the l dependence of ψ_l comes from the factors $B_{l+j,l-j}(\lambda_j)$ and w_j^l.

The l dependence of $B_{l+j,l-j}(\lambda_j)$ is determined from the second of the equations (3.27). We have

$$\begin{aligned} B_{l+j,l-j}(\lambda_j) &= b_+a_+\exp[i(\lambda_j + s + \gamma j)]A + b_+^2 \exp[-i(\lambda_j + s + l\gamma)]B \\ &\quad - a_+^2 \exp[i(\lambda_j + s + l\gamma)]C - b_+a_+ \exp[i(\lambda_j + s - \gamma j)]D \end{aligned} \tag{3.41}$$

where a_+ and b_+ are independent of l, and A, B, C, D are the components of the monodromy matrix (3.1). On the other hand, w_j^l is given by (3.12) (the equation corresponding to the sign +) and (2.13), i.e.

$$w_j^l = \begin{pmatrix} a_- \, e^{i/2}((l+j)\gamma + t) \\ b_- \, e^{-i/2}((l+j)\gamma + t) \end{pmatrix} \tag{3.42}$$

with a_- and b_- independent of l.

After replacing (3.42) in (3.19) and (3.41) in (3.32), the dependence on l of ψ_l follows easily. The components of this vector are linear combinations of terms of the form $e^{ip\gamma l}$, where p takes the values $-N/2 \leq p \leq N/2$. In consequence, taking into account (3.39), and after performing the sum over all the integer values of l, the resulting expression of ψ_ρ is a linear combination of terms of the form $\delta(\rho - p\gamma)$, where $\delta(x)$ is Dirac's delta function. Therefore, in order to have non-zero eigenvectors, the parameter ρ must be take the values

$$\rho = p\gamma \qquad -N/2 \leq p \leq N/2. \tag{3.43}$$

The allowed values of ρ proposed in [3] yield to equation (3.43) in the trigonometric limit. Therefore, our proof supports the arguments of [3] for the elliptic case. In the six-vertex limit, this parametrisation becomes trigonometric, and it is this mathematical simplication that enables us to obtain the allowed values of ρ in equation (3.39).

The connection between the present construction of the six-vertex eigenvectors and the usual one is a hard mathematical problem. It will be treated in a subsequent paper [20].

Up to now, we have considered generic values of the parameter γ in the discussion. Let us investigate the special case when γ is 2π times a rational number

$$\gamma = 2\pi Q'/Q. \tag{3.44}$$

Here Q' and Q are positive integers. In this case all the objects previously introduced become periodic or antiperiodic functions of their discrete indices, with period Q. That is

$$\begin{aligned} X_{l+Q} &= (-1)^Q X_l \\ Y_{l+Q} &= (-1)^{Q'} Y_l \\ w_{k,\pm}^{l+Q} &= (-1)^{Q'} w_{k,\pm}^l \\ M_{k+l+Q}^{\pm} &= (-1)^{Q'} M_{k+l}^{\pm} \\ \Omega_{\pm}^{l+Q} &= \Omega_{\pm}^l. \end{aligned} \tag{3.45}$$

Now, the restriction $2n = N$ (equation (3.33)) is relaxed. We see that in order to have an eigenvector it is enough that

$$2n = N + PQ \qquad \text{where} \qquad P \in \mathbb{Z}. \tag{3.46}$$

Then, we find from (3.20) and (3.45)

$$A_{l+n,l-n}(\theta)\Omega_N^{l-N} = (-1)^{Pm} \sin^N(\theta+\gamma)\Omega_N^{l-N-1} \tag{3.47}$$

and an analogous formula for $D_{l+n,l-n}$. In addition, (3.45) shows that in the sum (3.38) we can restrict l to a period Q. Then, ρ must be such that

$$e^{iQ\rho} = 1$$

in order that $\psi_\rho(\lambda_1, \ldots, \lambda_n)$ be an eigenvector. That is

$$\rho = \frac{2\pi q}{Q} \qquad q = 0, \ldots, Q-1$$

and

$$\phi_q(\lambda_1, \ldots, \lambda_n) = \sum_{l=0}^{Q-1} \exp\left(\frac{2\pi \mathrm{i} q l}{Q}\right) \psi_l(\lambda_1, \ldots, \lambda_n). \tag{3.48}$$

In the present case (3.43) requires

$$q = sQ' (\bmod Q) \qquad \text{with} \qquad |s| < N/2. \tag{3.49}$$

Therefore the allowed values of q are those integers between 0 and $Q-1$ fulfilling also (3.49).

The BAE are now written

$$N\phi(\mu_j, \gamma/2) = \sum_{l=1}^{n} \phi(\mu_j - \mu_l, \gamma) + \frac{4\pi q}{Q} + 2\pi I_j \qquad 1 \leqslant j \leqslant n \qquad 0 \leqslant q \leqslant Q-1 \tag{3.50}$$

where $\gamma = 2\pi m/Q$, the I_j are half-integers and $2n = N (\bmod Q)$.

In the generic case (3.40) the BAE are

$$N\phi(\mu_j, \gamma/2) = \sum_{l=1}^{N/2} \phi(\mu_j - \mu_l, \gamma) + 2\rho + 2\pi I_j \tag{3.51}$$

where

$$\phi(z, \alpha) = \mathrm{i} \log \frac{\sinh(z + \mathrm{i}\alpha)}{\sinh(z - \mathrm{i}\alpha)} \tag{3.52}$$

both in (3.49) and (3.50).

Eigenvectors of the *XXZ* Hamiltonian of the type (3.48) have been found in [12] using the coordinate Bethe ansatz for the special case (3.44).

Equations (3.40) are a system of algebraic equations in the variables $x_j \equiv e^{2\mu_j}(1 \leqslant j \leqslant N/2)$ of degree $3N/2$ for all $\rho \neq 0$. Therefore, they possess a richer set of roots than the usual BAE. We have found in simple examples that the states given by (3.32)-(3.40) reproduce all eigenvectors of $\tau(\theta)$. Besides, the eigenvectors associated to the extra roots of (3.40) (absent in the usual BAE) vanish. This suggests that the $\tau(\theta)$ eigenspace corresponding to the ordinary BA and our construction (equations (3.32)-(3.40)) are the same. A detailed analysis is reported in [20].

We want to stress that BAE like (3.40) appear in different frameworks. They differ from the usual BAE in the extra phase ρ which is j independent. One finds those BAE when eigenvectors of modified six-vertex transfer matrices [13]

$$\tau_\rho(\theta) = e^{\mathrm{i}\rho} A(\theta) + e^{-\mathrm{i}\rho} D(\theta) \tag{3.53}$$

are calculated using the usual Bethe ansatz (common eigenstates of $\tau_\rho(\theta)$ and S_z). This transfer matrix $\tau_\rho(\theta)$ can be interpreted as a vertex model with quasiperiodic boundary conditions

$$\sigma^x_{N+1} \pm \mathrm{i}\sigma^y_{N+1} = e^{\pm \mathrm{i}\rho}(\sigma^x_1 \pm \mathrm{i}\sigma^y_1). \tag{3.54}$$

Furthermore, BAE like (3.40) follow when local gauge transformations are performed in the *XXZ* chain [14]. Moreover, we derive (3.40) in § 3 in a still different physical situation. They describe a new basis of eigenvectors of the usual transfer matrix

$$\tau(\theta) = A(\theta) + B(\theta).$$

This $\tau(\theta)$ corresponds to *periodic boundary conditions.* We want to stress the remarkable fact that the same set of (3.40) describe very different problems.

In the case of twisted boundary conditions the phase ρ is fixed precisely by the chosen boundary conditions (3.54). In contrast, in our eigenvector construction the phase ρ varies, labelling the different sectors of the PBC $\tau(\theta)$ eigenspace. Now $\rho = p\gamma$ with $|p| < N/2$ and $p \in \mathbb{Z}$. The integer p labels here different sets of eigenvectors of $\tau(\theta)$. That is, p characterises sectors in the eigenvector space of $\tau(\theta)$. The ground state belongs to the $p = 0$ sector. In the $N = \infty$ limit the effect of p disappears from the eigenvalues.

4. The connection between vertex models and SOS models

Let us consider a vertex model in a rectangular lattice $N \times M$ with vertex weights as in figure 2. The partition function for periodic boundary conditions can be written as

$$Z_{NM}^{\text{vertex}}(u-v) = \sum_{\boldsymbol{\alpha},\boldsymbol{\gamma}} Z(\boldsymbol{\gamma\alpha}|\boldsymbol{\alpha\gamma}|u-v) \tag{4.1}$$

where $u - v$ is the spectral parameter, $\boldsymbol{\alpha} \equiv (\alpha_1, \dots, \alpha_N)$, $\boldsymbol{\gamma} \equiv (\gamma_1, \dots, \gamma_M)$ and

$$Z(\boldsymbol{\gamma}, \boldsymbol{\alpha}|\boldsymbol{\alpha}', \boldsymbol{\gamma}'|\theta) = \sum_{\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \dots, \boldsymbol{\lambda}_{M-1}} \prod_{i=1}^{M} T_{\gamma_i\gamma_i'}(\theta)_{\boldsymbol{\lambda}_i|\boldsymbol{\lambda}_{i-1}} \tag{4.2}$$

corresponds to the partition function of the vertex model with fixed boundary conditions, specified by α', γ', α and γ, as in figure 3. In (4.2) we assume

$$\lambda_{0,j} = \alpha_j', \lambda_{M,j} = \alpha_j \qquad \text{for} \qquad 1 \leq j \leq N \qquad \text{and}$$

$$T_{\gamma\gamma'}(\theta)_{\boldsymbol{\mu}|\boldsymbol{\mu}'} \equiv \sum_{\sigma_1 \dots \sigma_{N-1}} \prod_{j=1}^{N} R^{\mu_j'\sigma_j}_{\sigma_{j-1}\mu_j}(\theta) \tag{4.3}$$

with $\sigma_0 \equiv \gamma$ and $\sigma_N \equiv \gamma'$. The matrix $T_{\gamma\gamma'}(\theta)_{\boldsymbol{\mu}|\boldsymbol{\mu}'}$ is the monodromy matrix associated to a single horizontal line of the lattice, as depicted in figure 4.

Let us now connect the partition function of the SOS model defined by the weights $W(l_1, l_2, l_3, l_4|\theta)$ (figure 1) with Z_{NM}^{vertex} when the SOS weights relate to the R matrix through the intertwining vectors (equation (2.20)). We have for the SOS partition function with periodic boundary conditions

$$Z_{NM}^{\text{SOS}}(u-v) = \sum_{\substack{\boldsymbol{l} \in \mathbb{Z}^N \\ \boldsymbol{m} \in \mathbb{Z}^M}} Z(\boldsymbol{ml}|\boldsymbol{lm}|u-v) \tag{4.4}$$

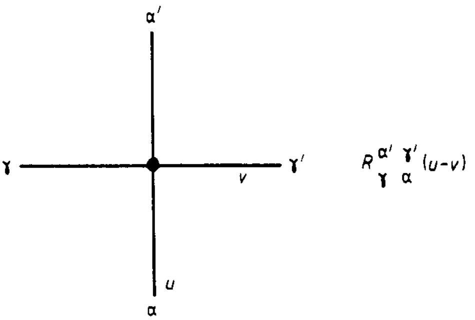

Figure 2. The arrangement of the bond variables round a vertex of the lattice and the elements of the R matrix.

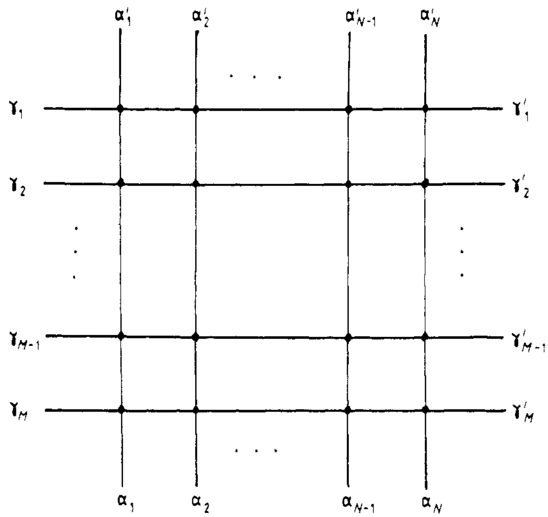

Figure 3. A vertex model formulated on a finite $N \times M$ square lattice with fixed boundary conditions, specified by the set of bond variables γ, γ', α, α'.

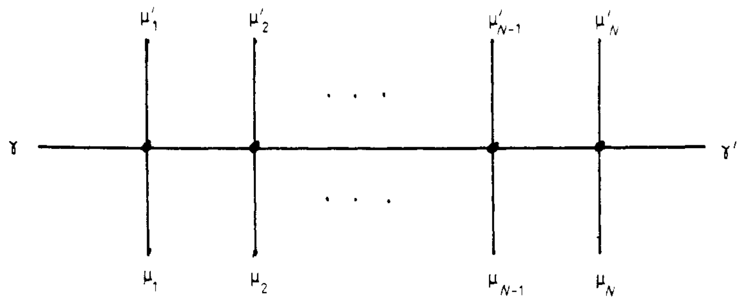

Figure 4. A row of the square lattice showing the variables associated with the various edges. The fixed boundary conditions in the horizontal direction are specified by the edge variables γ and γ'.

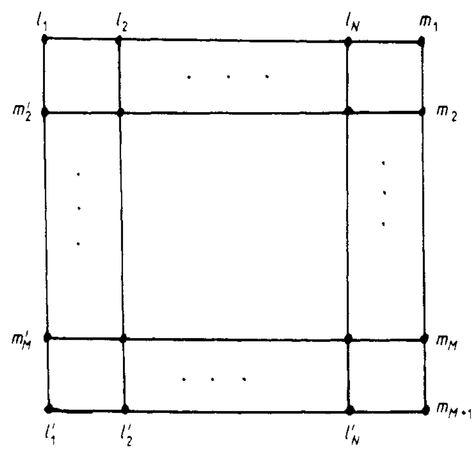

Figure 5. An SOS model formulated on a finite $N \times M$ square lattice with fixed boundary conditions, specified by the set of spin variables l, l', m, m'.

where $\boldsymbol{l} \equiv (l_1, \ldots, l_N)$, $\boldsymbol{m} = (m_1, \ldots, m_M)$ and $Z(\boldsymbol{m}'\boldsymbol{l}'|\boldsymbol{lm}|u-v)$ stands for the sos partition function with fixed boundary conditions as depicted in figure 5. It can be written as

$$Z(\boldsymbol{m}', \boldsymbol{l}'|\boldsymbol{lm}|\theta) = \prod_{j=1}^{M} \tau(\boldsymbol{p}_{j+1}|\boldsymbol{p}_j|\theta) \tag{4.5}$$

where $\boldsymbol{p}_1 \equiv \boldsymbol{l}$, $\boldsymbol{p}_{M+1} \equiv \boldsymbol{l}'$, $p_{i,1} = m'_i$, $p_{i,N+1} = m_i$ for $1 \leq i \leq M$ and $\tau(\boldsymbol{p}|\boldsymbol{p}'|\theta)$ is the sos transfer matrix defined by

$$\tau(\boldsymbol{p}'|\boldsymbol{p}|\theta) = \prod_{j=1}^{N} W(p'_j, p'_{j+1}, p_{j+1}, p_j|\theta) \tag{4.6}$$

where p' labels the lower line of faces and p the upper one, as in figure 6.

In all cases the indices associated with neighbouring faces differ by plus or minus one.

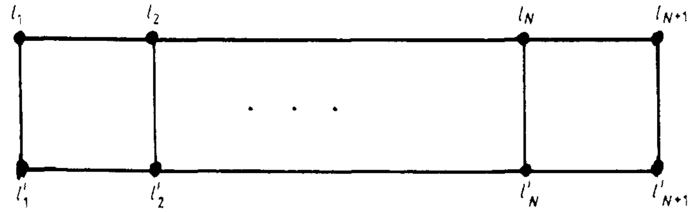

Figure 6. Two successive rows of the square lattice showing the spin variables associated with the various sites.

In order to relate $Z_{NM}(\theta)^{\text{vertex}}$ with $Z_{NM}(\theta)^{\text{SOS}}$ let us apply $Z(\gamma\alpha|\alpha'\gamma'|u-v)$ to the following tensor product of intertwining vectors:

$$\sum_{\boldsymbol{\alpha}',\boldsymbol{\gamma}'} Z(\boldsymbol{\gamma\alpha}|\boldsymbol{\alpha}'\boldsymbol{\gamma}'|u-v) X^{(\boldsymbol{m})}(v|\boldsymbol{\gamma}') X^{(\boldsymbol{l})}(u|\boldsymbol{\alpha}'). \tag{4.7}$$

Here,

$$\begin{aligned} X^{(\boldsymbol{l})}(u|\boldsymbol{\alpha}) &\equiv X^{l_1 l_2}_{\alpha_1}(u) X^{l_2 l_3}_{\alpha_2}(u) \ldots X^{l_N, l_{N+1}}_{\alpha_N}(u) \\ X^{(\boldsymbol{m})}(v|\gamma) &\equiv X^{m_1, m_2}_{\gamma_1}(v) X^{m_2 m_3}_{\gamma_2}(v) \ldots X^{m_M, m_{M+1}}_{\gamma_M}(v) \end{aligned} \tag{4.8}$$

where

$$l_{N+1} \equiv m_1 \qquad \text{and} \qquad |l_i - l_{i+1}| = |m_i - m_{i+1}| = 1$$

as usual. The sum over α' and γ' can be performed by repeatedly using (2.19). In this way at each vertex of figure 2, the sum over two bond indices yields a sum over a face index of weights w. We finally obtain

$$\begin{aligned} &\sum_{\boldsymbol{\alpha}',\boldsymbol{\gamma}'} Z(\boldsymbol{\gamma\alpha}|\boldsymbol{\alpha}'\boldsymbol{\gamma}'|u-v) X^{(\boldsymbol{m})}(v|\boldsymbol{\gamma}') X^{(\boldsymbol{l})}(u|\boldsymbol{\alpha}') \\ &\qquad = \sum_{\boldsymbol{m}',\boldsymbol{l}'} Z(\boldsymbol{m}'\boldsymbol{l}'|\boldsymbol{lm}|u-v) X^{(\boldsymbol{m}')}(v|\boldsymbol{\gamma}) X^{(\boldsymbol{l}')}(u|\boldsymbol{\alpha}) \end{aligned} \tag{4.9}$$

where

$$l'_{N+1} \equiv m_{M+1} \qquad m'_{M+1} \equiv l'_1.$$

Now, in the last step we use the orthogonality relations for the intertwining vectors (see § 2)

$$\begin{aligned}&\hat{X}^{l',l}(u)X^{l'',l}(u)=\delta_{l'l''}\\&\hat{X}^{l,l'}(u)X^{l,l''}(u)=\delta_{l'l''}\end{aligned}\tag{4.10}$$

where

$$\hat{X}^{l,l\pm\varepsilon}(u)\equiv\frac{\tilde{X}^{l,l\mp\varepsilon}(u)}{\Delta^{l,l\mp\varepsilon}(u)}\tag{4.11}$$

and $\Delta^{l,l'}(u)=(l-l')\Delta(u)$ with $|l-l'|=1$ and

$$\Delta(u)=a_+b_-\exp\{\mathrm{i}[u+(s-t)/2]\}-a_-b_+\exp\{-\mathrm{i}[u+(s-t)/2]\}.\tag{4.12}$$

By analogy with (4.9) we define

$$\begin{aligned}&\hat{X}^{(m')}(v|\boldsymbol{\gamma})=\hat{X}_{\gamma_1}^{m_1',m_2'}(v)\hat{X}_{\gamma_2}^{m_2',m_3'}(v)\ldots\hat{X}_{\gamma_M}^{m_M',m_{M+1}'}(v)\\&\hat{X}^{(l')}(u|\boldsymbol{\alpha})=\hat{X}_{\alpha_1}^{l_1',l_2'}(u)X_{\alpha_2}^{l_2',l_3'}(u)\ldots X_{\alpha_N}^{l_N',l_{N+1}'}(u).\end{aligned}\tag{4.13}$$

Finally, we get from (4.9) using (4.10) and (4.13)

$$Z(\boldsymbol{m}'\boldsymbol{l}'|\boldsymbol{l}\boldsymbol{m}|u-v)=\sum_{\boldsymbol{\alpha\alpha'\gamma\gamma'}}\hat{X}^{(m')}(v|\boldsymbol{\gamma})\hat{X}^{(l')}(u|\boldsymbol{\alpha})Z(\boldsymbol{\gamma\alpha}|\boldsymbol{\alpha}'\boldsymbol{\gamma}'|u-v)X^{(m)}(v|\boldsymbol{\gamma}')X^{(l)}(u|\boldsymbol{\alpha}').\tag{4.14}$$

This is the relation between SOS and vertex partition functions for fixed boundary conditions. Setting $l=l'$, $m=m'$ and summing over l and m in (4.14) yields

$$Z_{NM}^{\mathrm{SOS}}(u-v)=\sum_{lm}\sum_{\boldsymbol{\alpha},\boldsymbol{\alpha}',\boldsymbol{\gamma},\boldsymbol{\gamma}'}\hat{X}^{(m)}(v|\boldsymbol{\gamma})\hat{X}^{(l)}(u,\boldsymbol{\alpha})Z(\boldsymbol{\gamma\alpha}|\boldsymbol{\alpha}'\boldsymbol{\gamma}'|u-v)X^{(m)}(v|\boldsymbol{\gamma}')X^{(l)}(u,\boldsymbol{\alpha}').\tag{4.15}$$

We want to stress that (4.14) and (4.15) follow just from the fundamental relation between the R matrix and the SOS weights (2.19), and the orthogonality relations (4.10). They hold also for more general models like the eight-vertex, the Belavin model and others [2]. Equation (4.15) has also been obtained in [17]. As can be seen from (4.15), the relation between the partition function of the SOS model with periodic boundary conditions, and the partition function of the corresponding vertex model is, at first sight, very complicated.

On the right-hand side of (4.15), the boundary conditions of the vertex model are very unusual. A cumbersome procedure, involving products and sums with intertwining vectors, must be performed in order to obtain, on the left-hand side, the partition function of the SOS model with periodic boundary conditions.

However, if we impose other types of boundary conditions on the SOS model, a much more simple relation between the partition functions is obtained. We will impose periodic boundary conditions for all the boundary spins of the SOS model on an $N\times M$ lattice, with the exception of the four spins in the corners of the lattice.

As can be seen from figure 5, we will have in that case

$$\begin{aligned}&l_2=l_2',\ l_3=l_3',\ldots,\ l_N=l_N'\\&m_2=m_2',\ m_3=m_3',\ldots,\ m_M=m_M'\\&l_1\neq m_1\qquad m_1\neq m_{M+1}\\&l_1\neq l_1'\qquad l_1'\neq m_{M+1}.\end{aligned}\tag{4.16}$$

Now, taking into account (4.16), we perform the sum in equation (4.14) over all the boundary spins m, m', l and l', except for the four spins l_1, m_1, l'_1 and m_{M+1}.

On the left-hand side we obtain the partition function of the SOS model $Z(l_1, m_1, l'_1, m_{M+1})$, with the almost-periodic boundary conditions described above, and with fixed spins l_1, l'_1, m_1, m_{M+1} on the corners. Of course, the configurations of the boundary spins must satisfy the condition

$$|l_i - l_j| = 1 \tag{4.17}$$

where i, j indicates a pair of nearest-neighbour sites in the horizontal or vertical directions. On the right-hand side of (4.14), a great simplification occurs when the sum described above is performed.

Let us consider, in particular, the sum over two nearest-neighbour spins l_i and l_{i+1} on the boundary of the lattice and with $i \neq 1$. For each pair l_i, l_{i+1} there are four factors in (4.14) containing these spin variables.

Let us exhibit these factors explicitly:

$$\sum_{l_i, l_{i+1}} X_{\alpha'i}^{l_{i-1}, l_i}(u) X_{\alpha'_{i+1}}^{l_i, l_{i+1}}(u) \hat{X}_{\alpha_i}^{l_{i-1}, l_i}(u) \hat{X}_{\alpha_{i+1}}^{l_i, l_{i+1}}(u). \tag{4.18}$$

Where we have taken into account the expressions given in (4.8) and (4.13). The condition (4.17) enables us to introduce the variables τ_{i-1} and τ_i, which take the values -1 and 1, as follows:

$$l_i - l_{i-1} = \sigma_{i-1} \qquad\qquad l_{i+1} - l_i = \sigma_i.$$

In this way, (4.18) can be written as

$$\sum_{\sigma_{i-1}, \sigma_i} X_{\alpha'_i}^{l_{i-1}, l_{i-1}+\sigma_{i-1}}(u) X_{\alpha'_i+1}^{l_i, l_i+\sigma_i}(u) \hat{X}_{\alpha_i}^{l_{i-1}, l_{i-1}+\sigma_{i-1}}(u) \hat{X}_{\alpha_{i+1}}^{l_i, l_i+\sigma_i}(u). \tag{4.19}$$

As can be easily verified from the definition of the M matrix (3.4), we have

$$M_{\alpha\beta}(u) = X_{\alpha}^{l, l+\beta}(u) \qquad\qquad [M^{-1}]_{\alpha\beta}(u) = \hat{X}_{\beta}^{l, l+\alpha}(u) \tag{4.20}$$

where α, $\beta = \pm 1$ and M is the 2×2 matrix defined in (3.4). Therefore, (4.19) can be expressed as

$$\sum_{\sigma_{i-1}, \sigma_i} M_{\alpha'_i, \sigma_{i-1}}(u) M_{\alpha'_{i+1}, \sigma_i}(u) [M^{-1}]_{\sigma_{i-1}, \alpha_i}(u) [M^{-1}]_{\sigma_i, \alpha_{i+1}}(u) = \delta_{\alpha'_i, \alpha_i}. \tag{4.21}$$

When one of the two nearest-neighbour spins is in a corner of the lattice, the result of this type of sum is different from (4.21). If, for instance, $i - 1 = 1$ we have

$$\sum_{l_2, l_3} X_{\alpha'_1}^{l_1, l_2}(u) X_{\alpha'_2}^{l_2, l_3}(u) \hat{X}_{\alpha_1}^{l'_1, l_2}(u) \hat{X}_{\alpha_2}^{l_2, l_3}(u). \tag{4.22}$$

Now, by introducing the variables

$$\sigma_1 = l_2 - l_1 \qquad\qquad \sigma_2 = l_3 - l_2$$

(4.22) can be written as

$$\sum_{\sigma_1, \sigma_2} X_{\alpha'_1}^{l_1, l_1+\sigma_1}(u) X_{\alpha'_2}^{l_2, l_2+\sigma_2}(u) \hat{X}_{\alpha_1}^{l'_1, l_1+\sigma_1}(u) \hat{X}_{\alpha_2}^{l_2, l_2+\sigma_2}(u). \tag{4.23}$$

Using (4.20), the sum over τ_2 can be performed to give a factor $\delta_{\alpha'_2, \alpha_2}$. However, as a consequence of the fact that $l_1 \neq l'_1$, there is no such simplification in the sum over τ_1.

In summary, (4.23) yields to

$$\delta_{\alpha_2',\alpha_2} \sum_{\sigma_1} X_{\alpha_1'}^{l_1,l_1+\sigma_1}(u) \hat{X}_{\alpha_1}^{l_1',l_1+\sigma_1}(u). \tag{4.24}$$

Similar results are obtained for the other corners of the lattice. It must be stressed that in order to have the simplifications occurring in (4.21) it is not possible to impose periodic boundary conditions. In that case we would have, for the first row of figure 5, with $m_1 = l_1$

$$\begin{aligned} \sigma_1 &= l_2 - l_1 \\ \sigma_2 &= l_3 - l_2 \\ &\vdots \\ \sigma_{N-1} &= l_N - l_{N-1} \end{aligned} \tag{4.25}$$

but the variable $\tau_N = l_1 - l_N$ is not independent. It is fixed by the relations (4.25). In consequence, the equation (4.21) cannot be applied in that case.

This is the reason why we have chosen the almost-periodic boundary conditions (4.16). Taking into account (4.21) and (4.24) for each corner of the lattice, the equation obtained from (4.14) after the sum over all the boundary spins with the exception of the corner ones, is

$$\begin{aligned} Z_{\mathrm{SOS}}(l_1, l_1', m_1, m_{M+1}) &= \sum_{l_2, m_2, l_N, m_M} X_{\alpha_1'}^{l_1,l_2} \hat{X}_{\alpha_1}^{l_1',l_2} X_{\gamma_1'}^{m_1,m_2} \hat{X}_{\gamma_1}^{l_1,m_2} X_{\gamma_M'}^{m_M,m_{M+1}} \hat{X}_{\gamma_M}^{m_M,l_1'} \\ &\quad \times X_{\alpha_N'}^{l_N,m_1} \hat{X}_{\alpha_N}^{l_N,m_{M+1}} Z_{\mathrm{vertex}}(\gamma_1, \gamma_1', \gamma_M, \gamma_M', \alpha_1, \alpha_1', \alpha_N, \alpha_N'). \end{aligned} \tag{4.26}$$

This relation between the partition functions of both models is simpler by far than (4.15). In (4.15) the number of intertwining vectors that appear in the product of the right-hand side is $2(N+M)$. In contrast, in (4.26) this number is 8, independently of the size of the lattice. In particular, the partition function of the vertex model on the right-hand side of (4.26) has also periodic boundary conditions, with the exception of the corners of the lattice. Let us now consider the case $M \to \infty$ and N large but finite. In the evaluation of the finite-size corrections of the eigenvalues of both models, the different types of boundary conditions in the corners of lattice will give a contribution much smaller than $1/N^2$. This yields the conformal properties of the models. Therefore, as consequence of equation (4.26), the conformal properties like the central charge of the vertex model and the associated SOS model will be the same for arbitrary values of the parameter γ.

5. SOS models and the six-vertex model; charge projector formalism

In § 2 we used intertwining vectors to derive an IRF model from the six-vertex model. The weights thus obtained were l independent and actually identical to those of the six-vertex model. We shall see now how richer IRF models arise from the six-vertex model, taking into account the presence of the U(1)-conserved charge in the six-vertex model. We recall that the eight-vertex model enjoys a discrete $Z_2 \otimes Z_2$ symmetry that enlarges to $\mathrm{U}(1) \otimes Z_2$ in the critical limit.

The six-vertex R matrix commutes with the charge projectors P_Q

$$[R(\theta), P_Q] = 0 \qquad Q = 0, \pm 1 \tag{5.1}$$

where

$$P^{+1}=\begin{pmatrix}1 & \\ & 0\end{pmatrix} \qquad P_{-1}=\begin{pmatrix}0 & \\ & 1\end{pmatrix} \qquad P_0=\begin{pmatrix}0 & & & 0\\ & 1 & 0 & \\ & 0 & 1 & \\ 0 & & & 0\end{pmatrix}. \tag{5.2}$$

Therefore one can look for intertwining vectors in each charged sector

$$R(u-v)P_Q[X_l^Q(u)\otimes X_{l+1}^Q(v)]=\lambda(u-v)P_Q[X_l^Q(v)\otimes X_{l+1}^Q(u)] \tag{5.3}$$

where $X_l^Q(u)$ are two-component vectors. For $Q=\pm 1$, equation (5.3) has the solutions

$$\begin{aligned} X_l^{+1}(u)&=\begin{pmatrix}1\\0\end{pmatrix} \qquad Q=+1\\ X_l^{-1}(u)&=\begin{pmatrix}0\\1\end{pmatrix} \qquad Q=-1 \end{aligned} \tag{5.4}$$

with the same IRF weight

$$\lambda(u-v)=\sin(u-v+\gamma). \tag{5.5}$$

The case $Q=0$ has richer structure. We find the solution

$$X_l(u)=\begin{pmatrix}\sin(u-l\gamma-t)\\ 1\end{pmatrix} \tag{5.6}$$

with the same IRF weight (5.5). In addition, the vector

$$Y_k(u)=\begin{pmatrix}\sin(u+k\gamma+s)\\ 1\end{pmatrix} \tag{5.7}$$

fulfils

$$R(u-v)P_0[Y_{l+1}(u)\otimes Y_l(v)]=\sin(u-v+\gamma)R_0[Y_{l+1}(u)\otimes Y_l(v)]$$

where s and t are arbitrary parameters in (5.6) and (5.7). More generally, $X_l(u)$ and $Y_k(v)$ obey

$$\begin{aligned} &R(u-v)P_0[X_k(u)\otimes Y_l(v)]\\ &\qquad =\alpha_{k+l}(u-v)P_0[X_k(v)\otimes Y_l(u)]+\beta_{k+l}(u-v)P_0[Y_{l-1}(v)\otimes X_{k-1}(u)]\\ &R(u-v)P_0[Y_l(u)\otimes X_k(v)]\\ &\qquad =\bar{\alpha}_{k+l}(u-v)P_0[Y_l(v)\otimes X_k(u)]+\bar{\beta}_{k+l}(u-v)P_0[X_{k+1}(v)\otimes Y_{l+1}(u)] \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} \alpha_{k+l}(\theta)&=-\frac{\sin(\gamma)\{\sin\theta+\sin[\theta-(l+k-1)\gamma-s-t]\}}{\sin[(l+k-1)\gamma+s+t]}\\ \beta_{k+l}(\theta)&=\frac{\sin(\theta)\{\sin(\gamma)+\sin[(k+l)\gamma+s+t]\}}{\sin[(l+k-1)\gamma+s+t]}\\ \bar{\alpha}_{k+l}(\theta)&=\frac{\sin(\gamma)\{\sin(\theta)+\sin[\theta+(l+k+1)\gamma+t+s]\}}{\sin[(l+k+1)\gamma+t+s]}\\ \bar{\beta}_{k+l}(\theta)&=\frac{\sin(\theta)\{\sin[(k+l)\gamma+s+t]-\sin(\gamma)\}}{\sin[(l+k+1)\gamma+s+t]}. \end{aligned} \tag{5.9}$$

Equations (5.6)-(5.8) can be recast in a canonical form analogous to (2.19):

$$R(u-v)P_0[X^{l,m}(u)\otimes X^{m,n}(v)] = \sum_p W(p,n,l,m|u-v)P_0[X^{l,p}(v)\otimes X^{p,n}(u)] \tag{5.10}$$

where

$$X^{l,l+1}(u)=X_l(u) \qquad X^{l,l-1}(u)=Y_l(u)$$

and the weights $W(p,n,l,m|\theta)$ are given by

$$\begin{aligned} &W(l,l\pm1,l\mp1,l)=\sin(\theta+\gamma) \qquad W(l-1,l,l,l+1)=\beta_{2l+1} \\ &W(l+1,l,l,l+1)=\alpha_{2l+1} \qquad W(l+1,l,l,l-1)=\tilde{\beta}_{2l-1} \\ &W(l-1,l,l,l-1)=\tilde{\alpha}_{2l-1}. \end{aligned} \tag{5.11}$$

In this way, an IRF model with l-dependent weights arises from the six-vertex R matrix. The weights (5.11) can be transformed, by gauge transformations, in the critical limit of the ABF (elliptic) weights (the partition function being the same). This gauge transformation can be expressed as

$$\bar{W}(l,m',l',m)=W(l,m',l',m)\frac{F(l,m')F(l,l')}{F(l',m)F(m',m)} \tag{5.12}$$

with

$$F(l,m)=F(m,l)$$

and

$$\frac{F(l,l+1)}{F(l,l-1)}=\left(\frac{\sin[(l+1)\gamma+(s+t)/2]}{\sin[(l-1)\gamma+(s+t)/2]}\right)^{1/4}. \tag{5.13}$$

This IRF model and the six-vertex model can also be connected by the q-analogue of the Clebsch-Gordan coefficients [18], and using appropriately generalised intertwining vectors [19]. We find from (5.11)-(5.13) the weights

$$\begin{aligned} &W(l,l\pm1,l\mp1,l|\theta)=\sin(\theta+\gamma) \\ &W(l\pm1,l,l,l\mp1|\theta)=\frac{\sin(\theta)\{\sin[(l+1)\gamma+w_0]\sin[(l-1)\gamma+w_0]\}^{1/2}}{\sin(l\gamma+w_0)} \\ &W(l\pm1,l,l,l\pm1|\theta)=\frac{\sin(\gamma)\sin[l\gamma+w_0\mp\theta]}{\sin(l\gamma+w_0)} \end{aligned} \tag{5.14}$$

where $w_0=(s+t)/2$. It is easy to check that (5.14) is the critical limit of the SOS and RSOS weights of ABF [8].

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